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Running wave solutions of the two-dimensional sine-Gordon equation

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Abstract. After the Lamb substitution the 2D sine-Gordon equation was solved. Three classes of solutions for this equation were found. *Ad hoc* the 2D sine-Gordon equation was reduced to an algebraic system consisting of three equations. The solutions given of the 2D sine-Gordon equation are a generalization of the solutions of the 1D sine-Gordon equation. They are also a generalization of the solutions of the equation $\phi_{xx}+\phi_{yy}=\sin\phi(x,y)$.

The 1D sine-Gordon equation

$$\phi_{xx}-\phi_{tt}=\sin\phi(x,t)$$

is well known. Now the importance of this equation is due to its physical applications [1] and to the fact that it has soliton solutions [2].

In this paper the 2D sine-Gordon equation

$$\phi_{xx}+\phi_{yy}-\phi_{tt}=\sin\phi(x,y,t) \tag{1}$$

is studied. It may be regarded as describing solitary waves in a 2D Josephson junction. A method for solving of equation (1) is presented.

As a result of the Lamb substitution [3]

$$\phi(x,y,t)=4\tan^{-1}[M(x,y,t)] \tag{2}$$

the mathematical form of the 2D sine-Gordon equation is as follows:

$$(1+M^2)(M_{xx}+M_{yy}-M_{tt})-2M[(M_x)^2+(M_y)^2-(M_t)^2]=M(1-M^2). \tag{3}$$

If

$$M(x,y,t)=T_1(x)T_2(y,t) \tag{4}$$

then the result is

$$\begin{aligned}(1+T_1^2T_2^2)&(T_2T_{1xx}+T_1T_{2yy}-T_1T_{2tt})\\&-2T_1T_2[(T_{1x})^2T_2^2+T_1^2(T_{2y})^2-T_1^2(T_{2t})^2]\\&=T_1T_2(1-T_1^2T_2^2).\end{aligned} \tag{5}$$

The nature of the functions $T_1(x)$ and $T_2(y,t)$ is studied. After elementary algebraic operations the mathematical form of equation (5) is as follows:

$$\begin{aligned}T_1^3[T_2^2T_{2yy}&-T_2^2T_{2tt}-2(T_{2y})^2T_2+2T_2(T_{2t})^2+T_2^3]\\&+T_1[T_{2yy}-T_{2tt}-T_2]+\mu(x,y,t)=0.\end{aligned} \tag{6}$$

Here the function $\mu(x, y, t)$ is

$$\mu(x, y, t) = T_{1xx}T_2 + T_{1xx}T_1^2T_2^3 - 2(T_{1x})^2T_1T_2^3. \tag{7}$$

The form of $\mu(x, y, t)$ compatible with (6) is

$$\mu(x, y, t) = T_1F(T_2) + T_1^3G(T_2) \tag{8}$$

$$F(T_2) = pT_2 + qT_2^3 \tag{9a}$$

$$G(T_2) = rT_2 + sT_2^3. \tag{9b}$$

Here p, q, r, s are parameters. If (8) is substituted into (7) the result is

$$T_2(T_{1xx} - pT_1 - rT_1^3) + T_2^3[T_{1xx}T_1^2 - 2(T_{1x})^2T_1 - qT_1 - sT_1^3] = 0. \tag{10}$$

Then $T_1(x)$ is such that

$$T_{1xx} - pT_1 - rT_1^3 = 0 \tag{11}$$

$$T_{1xx}T_1^2 - 2(T_{1x})^2T_1 - qT_1 - sT_1^3 = 0. \tag{12}$$

Equations (11) and (12) are compatible only when $s = -p$. They can be reduced to the equation

$$(T_{1x})^2 = \tfrac{1}{2}rT_1^4 + pT_1^2 - \tfrac{1}{2}q. \tag{13}$$

Here p, q, r are arbitrary parameters and the conclusion is that T_1 is an elliptic function.

The function $T_2(y, t)$ is also studied. If (9) is substituted into (8) the result is

$$\mu(x, y, t) = T_1(pT_2 + qT_2^3) + T_1^3(rT_2 + sT_2^3). \tag{14a}$$

But $s = -p$ and then

$$\mu(x, y, t) = T_1(pT_2 + qT_2^3) + T_1^3(rT_2 - pT_2^3). \tag{14b}$$

If (14*b*) is substituted into (6) the final result is

$$T_1^3[T_{2yy}T_2^2 - T_{2tt}T_2^2 - 2(T_{2y})^2T_2 + 2(T_{2t})^2T_2 + (1-p)T_2^3 + rT_2] + T_1[T_{2yy} - T_{2tt} + (p-1)T_2 + qT_2^3] = 0. \tag{15}$$

The consistency of this system is ensured if $T_2(y, t)$ is such that

$$T_{2yy}T_2^2 - T_{2tt}T_2^2 - 2(T_{2y})^2T_2 + 2(T_{2t})^2T_2 + (1-p)T_2^3 + rT_2 = 0 \tag{16a}$$

$$T_{2yy} - T_{2tt} + (p-1)T_2 + qT_2^3 = 0. \tag{16b}$$

This system of equations is equivalent to the system

$$2(T_{2y})^2 - 2(T_{2t})^2 - 2(1-p)T_2^2 + qT_2^4 - r = 0 \tag{17}$$

$$4T_{2y}T_{2yy} - 4T_{2y}T_{2tt} - 4(1-p)T_{2y}T_2 + 4qT_{2y}T_2^3 = 0. \tag{18}$$

If we differentiate equation (17) with respect to y and then subtract the result from equation (18) the result will be

$$T_{2y}T_{2tt} = T_{2t}T_{2ty}. \tag{19}$$

The solution of the equation

$$T_{2t} = \delta\vartheta T_{2y} \qquad \delta = \pm 1 \tag{20}$$

is also the solution of equation (19). ϑ is an arbitrary constant. The solution of equation (20) is

$$T_2(y, t) = T_2(y + \delta\vartheta t) \tag{21}$$

that is, $T_2(y, t)$ is a running wave. If

$$v = y + \delta\vartheta t \tag{22}$$

is substituted into (17) the result is

$$(\delta^2\vartheta^2 - 1)(T_{2v})^2 = \tfrac{1}{2}qT_2^4 - (1-p)T_2^2 - \tfrac{1}{2}r. \tag{23}$$

The conclusion is that $T_2(v)$ is an elliptic function.

Let

$$T_1(x) = A_1 f(\alpha x;\, k_1) \tag{24a}$$

$$T_2(v) = A_2 g(\beta v;\, k_2) \tag{24b}$$

where A_1, A_2, α, β are parameters and f and g are Jacobi elliptic functions. Their generation equations are as follows:

$$(f')^2 = a_1 f^4 + b_1 f^2 + c_1 \tag{25a}$$

$$(g')^2 = a_2 g^4 + b_2 g^2 + c_2 \tag{25b}$$

where a_1, a_2, b_1, b_2, c_1, c_2 are parameters and the prime denotes differentiation by the corresponding variable (αx or βv). Let $\gamma = \beta\vartheta$ and $A = A_1A_2$. After the substitution of relations (24) and (25) into (13) and (23) and after comparing the parameters p, q, r from the resulting equations, the following three relations can be found:

$$\alpha^2 b_1 - (\delta^2\gamma^2 - \beta^2)b_2 = 1 \tag{26a}$$

$$\alpha^2 a_1 + (\delta^2\gamma^2 - \beta^2)A^2 c_2 = 0 \tag{26b}$$

$$(\delta^2\gamma^2 - \beta^2)a_2 + \alpha^2 A^2 c_1 = 0. \tag{26c}$$

The form of the solution of the 2D sine-Gordon equation in this case is

$$\phi = 4\tan^{-1}[Af(\alpha x;\, k_1)g(\beta y + \delta\gamma t;\, k_2)]. \tag{27}$$

The coefficients a_1, b_1, c_1, a_2, b_2, c_2 depend on the concrete elliptic functions f and g. The coefficients a_1, b_1, c_1 depend on the value of the elliptic integral module k_1: $a_1 = a_1(k_1)$; $b_1 = b_1(k_1)$; $c_1 = c_1(k_1)$. The coefficients a_2, b_2, c_2 depend on the value of the elliptic integral module k_2: $a_2 = a_2(k_2)$; $b_2 = b_2(k_2)$; $c_2 = c_2(k_2)$. So the solution of the 2D sine-Gordon equation depends on seven constants A, α, β, δ, γ, k_1, k_2 among which the three relations (26) exist. As $\delta = +1$ or $\delta = -1$ the solution depends on three free parameters. The relations (26) are characteristic relations for the solution (27) of the 2D sine-Gordon equation.

After the substitution $\beta = 0$ the characteristic relations (26) and the solution (27) are reduced to the corresponding solution and relations of the 1D sine-Gordon equation.

If $\gamma = 0$ then the solution (27) and the relations (26) are reduced to the corresponding solution and relations of the equation

$$\phi_{xx} + \phi_{yy} = \sin\phi(x, y). \tag{28}$$

Equation (28) is studied in connection with the 2D whirl models in statistical mechanics [4].

Three classes of solutions of the 2D sine-Gordon equation exist:

(i) *Class A:* $\delta^2\gamma^2-\beta^2>0$. *In this case if* $\bar{\gamma}^2=\delta^2\gamma^2-\beta^2$ *then the characteristic* relations (26) are reduced to the relations

$$\alpha^2 b_1-\bar{g}\gamma^2 b_2=1 \tag{29a}$$

$$\alpha^2 a_1+\bar{\gamma}^2 A^2 c_2=0 \tag{29b}$$

$$\bar{\gamma}^2 a_2+\alpha^2 A^2 c_1=0. \tag{29c}$$

The relations (29) are similar to the relations of the 1D sine-Gordon equation. In such a case this class of solutions of the 2D sine-Gordon equation is a generalization of the solutions of the 1D sine-Gordon equation.

(ii) *Class B:* $\delta^2\gamma^2-\beta^2<0$. In this case if $\bar{\beta}^2=-(\delta^2\gamma^2-\beta^2)$ then the characteristic relations (26) are reduced to the relations

$$\alpha^2 b_1+\bar{\beta}^2 b_2=1 \tag{30a}$$

$$\alpha^2 a_1-\bar{\beta}^2 A^2 c_2=0 \tag{30b}$$

$$-\bar{\beta}^2 a_2+\alpha^2 A^2 c_1=0. \tag{30c}$$

The relations (30) are similar to the relations of equation (28). In such a case this class of solutions of the 2D sine-Gordon equation is a generalization of the solutions of equation (28).

(iii) *Class C:* $\delta^2\gamma^2-\beta^2=0$. Then the form of the system of equations (26) is

$$\alpha^2 b_1=1 \tag{31a}$$

$$a_1=0 \tag{31b}$$

$$c_1=0 \tag{31c}$$

and the solution of equation (25*a*) is as follows: $f(x)=\exp(\delta x)$, $\delta=1$ or $\delta=-1$. The function f is an exponential function. In this case a_2, b_2 and c_2 are arbitrary parameters and then the function g is an arbitrary Jacobi elliptic function. In such a case the form of the solution of the 2D sine-Gordon equation is

$$\phi=4\tan^{-1}\{A\exp(\delta x)g[\delta\gamma(y\pm t);\,k_2]\}. \tag{32}$$

Solution class C is generalized as follows: the function g is not only an arbitrary Jacobi elliptic function. In general the function g is an arbitrary function. In this case $T_1(x)=\exp(\delta x)$ and $T_2(y,t)=Ag[\delta\gamma(y\pm t)]$; $T_1(x)$ obeys equation (13). If $r=0$, $q=0$, $p=1$ then $T_1(x)=\exp(\delta x)$. $T_2(y,t)$ obeys the systeh of equations (17) and (18). If $r=0$, $q=0$, $p=1$ then the form of the system is

$$(T_{2y})^2-(T_{2t})^2=0 \tag{33a}$$

$$T_{2yy}-T_{2tt}=0. \tag{33b}$$

It is at once apparent that the function $T_2(y,t)=Ag[\delta\gamma(y\pm t)]$ when g is an arbitrary function is a solution of equations (33). Solution class C of the 2D sine-Gordon equation was found [5].

A method for solving the 2D sine-Gordon equation is presented in this paper. The solutions found describe the phase difference between the wavefunctions of the electrons in the superconductors of the Josephson junction. These solutions can describe the electric and magnetic fields in the permittivity layer of the junction. If appropriate boundary conditions are applied, solitons can exist in the permittivity layer. A detailed description of the classes of solutions of equation (1) will be presented in future papers.

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